

$2D$ – Fractional Supersymmetry for Rational Conformal Field Theory: Application for Third-Integer Spin States

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Abstract

A $2D$ – fractional supersymmetry theory is algebraically constructed. The Lagrangian is derived using an adapted superspace including, in addition to a scalar field, two fields with spins $1/3, 2/3$. This theory turns out to be a rational conformal field theory. The symmetry of this model goes beyond the super Virasoro algebra and connects these third-integer spin states. Besides the stress-momentum tensor, we obtain a supercurrent of spin $4/3$. Cubic relations are involved in order to close the algebra; the basic algebra is no longer a Lie or a super-Lie algebra. The central charge of this model is found to be $5/3$. Finally, we analyse the form that a local invariant action should take.

I. Introduction.

$2D$ - conformal invariance, after the work of Belavin, Polyakov and Zamolodchikov [1], becomes a formidable tool for the description of $2D$ - critical phenomena and string theory. In that context, a study of $2D$ conformal field constitutes a great challenge for a classification of integrable models, and a description of $4D$ -string (in order to obtain string solutions with a good phenomenology). The first attempt, in a systematic classification, has been done by Friedan, Qiu and Shenker [2] who argued that if one imposes unitarity and a spectrum bounded from below (highest weight representation) one gets, within the framework of the Virasoro algebra, constraints on the values of the central charges c and the conformal weights h . They have obtained two different kinds of integrable models: one with $c > 1$ (and with an infinite number of primary fields), and discrete series for $c < 1$. However, it has been proved that if we enlarge the symmetry of the $2D$ -manifold, other series do appear. For instance, with a $N = 1$ superconformal algebra, other integrable models can be described with $c < 3/2$ [3]. Meanwhile, if one extends the symmetry of the worldsheet of the string, in the framework of super Virasoro algebra, the critical dimension goes consequently from 26 to 10 [4]. If one takes a $N = 2$ super-Virasoro algebra, as the basic symmetry of the worldsheet, the critical dimension is then $D = 2$.

Nevertheless, the Virasoro algebra can be extended using the parafermions introduced by Fateev and Zamolodchikov [5], and leading to new series of exact solvable models [6]. Noting that those theories naturally contain a current of spin $\frac{K+4}{K+2}$ ($K = 2$ corresponds to the superconformal algebra), they have been applied in the context of string theory with a critical dimension $2 + \frac{16}{K}$ [7]. All those solutions have a common feature, they can be realized in terms of the coset construction of Goddard, Kent and Olive (GKO) [8] with appropriate Kac-Moody algebras [9]. Let us point out that the GKO construction can be applied with all kinds of affine Lie algebras. For instance, the W_n algebras [10], obtained in this approach, involving primary fields up to spin n , close in its universal enveloping algebra; and so cannot be defined as a Lie algebra.

In the present paper, we will follow another direction to extend the Virasoro algebra by introducing currents of fractional spin, which does not close through quadratic relations, as W_n algebras. Our starting point is, neither the parafermions, nor the GKO coset construction, but the interesting property of $1, 2, 3 D$ - spaces where the states are not in a representation of the permutation group but rather of the braid group [11]. (We can point out that the parafermions have also non trivial monodromy transformations). This situation, previously exploited in order to extend supersymmetry to fractional supersymmetry [12, 13, 14, 15], has been considered in $1D$ [12, 14, 15] where this symmetry can be seen as a F^{th} root of the time translation ∂_t ; or in $2D$ as a F^{th} root of conformal transformations. $F = 2$ corresponds to the usual supersymmetry. This procedure, has been already applied in the case of $1D$ fractional supersymmetry. It leads to a new equation acting on the states which are in the representation of the braid group [15]. The method adopted there is similar to the one leading to the Dirac equation in $1D$ using the supersymmetry [16].

Here, we study $2D$ - fractional supersymmetry, *i.e.* we extend the Virasoro algebra with a current of spin $\frac{1+F}{F}$. In addition to the scalar field we introduce fields of conformal weight $\frac{1}{F} \dots \frac{F-1}{F}$. It turns out that the fractional supersymmetry is a symmetry which connects states of fractional spin. For $F = 3$, the central charge is

found to be $c = 5/3$, proving that this construction is different from the one of Ref. [6], because the central charge is, $c = \frac{3K}{K+2}$ ($c = 2$ for $K = 4$). When $F = 3$, we have a conserved current of spin $4/3$ like in the case $K = 4$. So, the conformal weights of this construction cannot be obtained through the approaches detailed in Ref. [6]. As far as we know, no GKO approach of this model has been built up.

In the literature, a similar approach has been already obtained by Saidi *et al.* [17]. These authors have also introduced fractional spin for fractional supersymmetry using parafermions and involving non local Operator Product Expansion (O.P.E.). A more detailed analysis of this point will be given in Section IV.

This paper will be divided as follow: Sect. II is devoted to a description of the main results already obtained in $1D$. In Sect. III, we construct explicitly the $2D$ fractional supersymmetric lagrangian, introducing an adapted fractional superspace by help of Generalized Grassmann variables and its differential structure. In Sect. IV, we calculate the Green function and we propose a normal ordering prescription in order to apply of the Wick theorem. We also discuss the q -mutation relations of the modes of the fields. In Sect. V, we determine the algebraic structure of the basic fields on behalf of the OPE. Sect. VI. is devoted to build action beyond FSUSY, *i.e.* by gauging the global symmetries. Finally, in the conclusion, we give an outlook of these new algebras, and obtain new critical dimensions for string.

II. Summary of the Main Results of $1D$ – Fractional Supersymmetry.

Supersymmetry, which is the only non-trivial Z_2 extension of the Poincaré algebra [18, 19], can be naturally generalized to fractional supersymmetry [12, 13, 14, 15], as soon as the space-time dimension is smaller than 3, where alternative statistics are allowed. The algebraic structure of fractional supersymmetry (FSUSY), possesses a Z_F structure (here, in this paper, we will consider only the case $F = 3$) so the basic fields will be of graduation $0, \dots, F-1$, generalizing the concept of boson/fermion which are respectively of graduation $0, 1$ *i.e* even/odd with respect to Z_2 .

Let us recall the main results of this symmetry in one dimension (this symmetry has already been introduced in $1D$ as a global symmetry [12, 14] or as a local one [15]).

FSUSY is generated by H , the Hamiltonian (or the generator of time translation) and Q , the generator of the FSUSY transformations. The algebra fulfills

$$\begin{aligned} [Q, H] &= 0 \\ Q^3 &= -H, \end{aligned} \tag{1}$$

it is important to emphasize that the algebra (1) is neither a Lie algebra nor a super-algebra, because it closes through a cubic relation, and goes beyond the framework of the Coleman, Mandula [18] and Haag, Sohnius, Lopuszanski [19] theorems, which deal with Lie or super Lie algebras. It is interesting to notice that most results of supersymmetry and supergravity (see *e.g.* [20]) can be transposed easily within the framework of fractional supersymmetry (for more details see [12, 13, 14, 15]). This symmetry acts on an analogous of a superspace introduced in supersymmetry (SUSY). The time t is then extended to (t, θ) with θ a real generalized Grassmann variable ($\theta^3 = 0$) [21], instead of a Grassmann one. The introduction of ϵ and f , the parameters of the FSUSY

transformations and the time translation, leads to the transformations [14, 15]

$$\begin{aligned} t' &= t + q(\epsilon^2\theta + \epsilon\theta^2) - f \\ \theta' &= \theta + \epsilon, \end{aligned} \quad (2)$$

ϵ verifies $\epsilon^3 = 0$ and $\theta\epsilon = q\epsilon\theta$, with $q = \exp(\frac{2i\pi}{3})$. The q -mutation between the two variables ϵ and θ has four origins :

- it ensures that if $\epsilon^3 = \theta^3 = 0$ then $(\epsilon + \theta)^3 = 0$ [21];
- the time remains real after a FSUSY transformation;
- the FSUSY transformations commute with the covariant derivative (see after);
- the FSUSY transformations (ϵQ , see after) satisfy the Leibnitz rule [see Durand in [12]].

Next, we consider a real fractional superfield Φ in the scalar representation of the fractional superline

$$\Phi(t, \theta) = x(t) + q^2\theta\psi_2(t) + q^2\theta^2\psi_1(t), \quad (3)$$

where $x(t), \psi_1(t), \psi_2(t)$ are the extensions of the bosonic and fermionic fields. They satisfy $\psi_1^3 = \psi_2^3 = 0$, and their grade is respectively 0, 1 and 2. They are submitted to the q -mutation relations (postulated from their grade)

$$\begin{aligned} \theta\psi_1(t) &= q\psi_1(t)\theta \\ \theta\psi_2(t) &= q^2\psi_2(t)\theta \\ \psi_2(t)\psi_1(t) &= q\psi_1(t)\psi_2(t), \end{aligned} \quad (4)$$

it can be stressed that these relations are the only ones which are arbitrary, all the other follow naturally [15]. Using relations (2), we get easily the FSUSY transformations upon the fields

$$\begin{aligned} \delta_\epsilon x &= q^2\epsilon\psi_2 \\ \delta_\epsilon\psi_2 &= -q\epsilon\psi_1 \\ \delta_\epsilon\psi_1 &= \epsilon\dot{x}. \end{aligned} \quad (5)$$

To build the action, we need to recall some basic features on the derivation acting on generalized Grassmann variables. This structure, the q -deformed Heisenberg algebra, has been analyzed in [21] as well as its matrix representation [21, 22]. It admits in general $(F - 1)$ derivatives. In our particular case, the two derivatives are noted ∂_θ and δ_θ with the properties

$$\begin{aligned} \partial_\theta\theta - q\theta\partial_\theta &= 1 \\ \delta_\theta\theta - q^2\theta\delta_\theta &= 1 \\ \partial_\theta^3 &= 0 \quad \delta_\theta^3 = 0 \\ \partial_\theta\delta_\theta &= q^2\delta_\theta\partial_\theta. \end{aligned} \quad (6)$$

Then, let us consider the two basic objects Q and D , which represent respectively the FSUSY generator and the covariant derivative [12, 13, 14, 15]

$$\begin{aligned} Q &= \partial_\theta + q\theta^2\partial_t \\ D &= \delta_\theta + q^2\theta^2\partial_t. \end{aligned} \quad (7)$$

It can be checked explicitly that $D^3 = Q^3 = -\partial_t$ and $QD = q^2DQ$, and a direct calculation (using $\theta\epsilon = q\epsilon\theta$) proves that

$$\delta_\epsilon\Phi = \epsilon Q\Phi(t, \theta). \quad (8)$$

Using the fact that D q -mutes with Q we have $\delta_\epsilon D\Phi = D\delta_\epsilon\Phi$. Finally, arguing that the θ^2 component of Φ transforms like a total derivative, we can construct the action by taking the θ^2 part of the action built in the fractional superspace. In other words, using the results on integration upon generalized Grassmann variables [23] $\int d\theta = \frac{d^{n-1}}{d\theta^{n-1}}$ we obtain for $n = 2$

$$\begin{aligned} S &= -\frac{q^2}{2} \int dt d\theta \dot{\Phi} D\Phi \\ &= \int dt \left(\frac{\dot{x}^2}{2} + \frac{q^2}{2} \dot{\psi}_1 \psi_2 - \frac{q}{2} \dot{\psi}_2 \psi_1 \right). \end{aligned} \quad (9)$$

This action has been extended under a local invariant form, introducing two gauge fields, and leads, after quantization, to an equation generalizing the Dirac one. A formulation, invariant under general reparametrization, has been given by means of a curved fractional superline and an analogous of a superdeterminant [15].

III. 2D– Fractional Supersymmetry on Riemann surfaces.

Now we want to extend all those results to build an action in the complex plane (2D FSUSY was introduced in Ref. [13]). This 2D– space might be used for the description of some 2D– integrable models; or even should represent the symmetry of the world-sheet of some string theories. The first step is to construct different sets of generalized Grassmann algebra (GGA) with its associated differential structure.

It is crucial to note that to endow the GGA with a complex structure (two generalized Grassmann variables $\theta, \bar{\theta}$ with $\bar{\theta} = \theta^*$) is *clearly* incompatible with the q -mutation ($\theta\bar{\theta} = q\bar{\theta}\theta$)*. So we cannot, as it could have been expected at first sight, generalize the 1D case directly by introducing a complex generalized Grassmann variable. So, we have to consider an alternative construction.

Like in heterotic string [24], where z and \bar{z} are extended differently ($z \rightarrow (z, \theta)$ and \bar{z} remains unaffected), here, we associate to z and \bar{z} two *real* generalized Grassmann variables θ_L and θ_R . In other words, the construction acts separately onto the L -movers and R -movers.

*If we would assume $\theta\bar{\theta} = q\bar{\theta}\theta$, with θ and $\bar{\theta}$ two complex conjugated variables, we get $\theta\bar{\theta}^2 = q^2\bar{\theta}^2\theta$ on one hand. If we conjugate this equation we get $\theta^2\bar{\theta} = q\bar{\theta}\theta^2$ on the other hand. This last equation clearly contradicts the hypothesis.

Consider the generalized Grassmann variables θ_i and its two derivatives (see (6)) ∂_i and δ_i (i running from 1 to p). In the previous case, we had $p = 2$ and $\theta_1 = \theta_L$, $\theta_2 = \theta_R$. From

$$\theta_i \theta_j = q \theta_j \theta_i, \quad i < j, \quad (10)$$

the consistency of the algebra leads to the following relations

$$\begin{aligned} \partial_i \partial_j &= q \partial_j \partial_i, \quad i < j \\ \partial_i \theta_j &= q^{-1} \theta_j \partial_i, \quad i < j \\ \partial_j \theta_i &= q \theta_i \partial_j, \quad i < j. \end{aligned} \quad (11)$$

We have the same relations with $\delta_i \rightarrow \partial_i$ (in fact $\delta_i \equiv (\partial_i)^*$ see after). These relations have been already derived by Mohammadi in Ref. [12]. Alternative derivation through matrix realization of the algebra has been obtained in Ref. [15]. A third derivation using a commuting set of GGA and changing the statistics through a Klein transformation is detailed in the Appendix.

Returning to our heterotic extension of the complex plane, we can however define an automorphism of the algebra exchanging (z, θ_L) and (\bar{z}, θ_R) . The algebra defined in relations (6) and (10-11) is NEITHER stable under complex conjugation, NOR under the permutation of the θ 's indices (we note σ this permutation). However, it is stable under the composition of both

$$(AB)^{\star \circ \sigma} = A^{\star \circ \sigma} B^{\star \circ \sigma}.$$

With such an automorphism, $(z, \theta_L, \partial_L, \delta_L)$ is mapped onto $(\bar{z}, \theta_R, \delta_R, \partial_R)$ and *vice versa*, so we see that we have a connection between the right-handed and the left-handed part of the action. Stress that under this conjugation, ∂_L is exchanged with δ_R . The next point before the construction of the action, is to remark that $(\partial \theta^a)^* = \theta^a \delta$, where ∂ acts from the *right* and δ from the *left*, this can be seen directly on the matrix realization of the algebra [15] and from $\partial^* = \delta$, $\theta^* = \theta$.

If we set

$$\begin{aligned} D_L &= \delta_L + q^2 \theta_L^2 \partial_z \\ Q_L &= \partial_L + q \theta_L^2 \partial_z, \end{aligned} \quad (12)$$

respectively the covariant derivative and the FSUSY generator associated to the z -modes. We obtain under the $\star \circ \sigma$ conjugation the covariant derivative and the FSUSY generator of the \bar{z} -modes

$$\begin{aligned} D_R &= \partial_R + q \theta_R^2 \partial_{\bar{z}} \\ Q_R &= \delta_R + q^2 \theta_R^2 \partial_{\bar{z}}. \end{aligned} \quad (13)$$

A direct calculation proves that $D_L^3 = Q_L^3 = -\partial_z$ and $D_R^3 = Q_R^3 = -\partial_{\bar{z}}$, as in 1D. Following Azcárraga and Macfarlane [14], D_L (resp. D_R) acts from the left (resp. the right). Introduce the fractional superfield,

$$\begin{aligned} \Phi(z, \theta_L, \bar{z}, \theta_R) &= X(z, \bar{z}) + q^2 \theta_L \psi_{20}(z, \bar{z}) + q^2 \theta_L^2 \psi_{10}(z, \bar{z}) \\ &+ q^2 \theta_R \psi_{02}(z, \bar{z}) + \theta_L \theta_R \psi_{22}(z, \bar{z}) + q^2 \theta_L^2 \theta_R \psi_{12}(z, \bar{z}) \\ &+ q^2 \theta_R^2 \psi_{01}(z, \bar{z}) + q^2 \theta_L \theta_R^2 \psi_{21}(z, \bar{z}) + \theta_L^2 \theta_R^2 \psi_{11}(z, \bar{z}). \end{aligned} \quad (14)$$

The components ψ_{ab} ($X = \psi_{00}$) are of grade $a + b$ and satisfy, because of their grade,

$$\begin{aligned}\theta_L \psi_{ab} &= q^{a+b} \psi_{ab} \theta_L \\ \theta_R \psi_{ab} &= q^{a+b} \psi_{ab} \theta_R.\end{aligned}\tag{15}$$

Now we are ready to build the $2D$ - action S . With similar arguments as those used in $1D$, and with $D_L(D_R)$ acting from the left(right) we get

$$\begin{aligned}S &= q \int dz d\bar{z} d\theta_R d\theta_L [D_L \Phi(z, \theta_L, \bar{z}, \theta_R) \Phi(z, \theta_L, \bar{z}, \theta_R) D_R] \\ &= \int dz d\bar{z} [\partial_z X(z, \bar{z}) \partial_{\bar{z}} X(z, \bar{z}) \\ &\quad - q \partial_z \psi_{02}(z, \bar{z}) \psi_{01}(z, \bar{z}) + q^2 \partial_z \psi_{01}(z, \bar{z}) \psi_{02}(z, \bar{z}) \\ &\quad + q \psi_{20}(z, \bar{z}) \partial_{\bar{z}} \psi_{10}(z, \bar{z}) - q^2 \psi_{10}(z, \bar{z}) \partial_{\bar{z}} \psi_{20}(z, \bar{z}) \\ &\quad - q \psi_{11}(z, \bar{z}) \psi_{22}(z, \bar{z}) - q^2 \psi_{22}(z, \bar{z}) \psi_{11}(z, \bar{z}) \\ &\quad + \psi_{12}(z, \bar{z}) \psi_{21}(z, \bar{z}) + \psi_{21}(z, \bar{z}) \psi_{12}(z, \bar{z})].\end{aligned}\tag{16}$$

First, note that this action is a grade 0 number. Second, if we choose $\psi_{ab}^* = \psi_{ba}$ and also with the appropriate choice of the power of q , in the definition of Φ , we ensure that the Lagrangian is real. Solving the equations of motion, we see that:

- X admits a holomorphic and an antiholomorphic part;
- ψ_{10}, ψ_{20} are holomorphic;
- ψ_{01}, ψ_{02} are antiholomorphic;
- $\psi_{12}, \psi_{21}, \psi_{11}, \psi_{22}$ are auxiliary fields that vanish on-shell.

En the $1D$ case, no modes expansion of the fields are allowed (except in the path integral formalism where developments on the eigenvectors can be used). However, in $2D$ (and upper dimensions), nothing can be said on the q -mutation of the various fields, but only on the modes of their associated Laurent expansions (see (10-11)). We will come back to this point further.

Finally, let us introduce ϵ_L and ϵ_R the parameters of the FSUSY transformations. Utilizing

(1) the structure of the algebra, for the L and R handed sectors ($Q_L D_L = q^2 D_L Q_L$ and $Q_R D_R = q D_R Q_R$), and from the fact that the covariant derivative has to commute with the FSUSY transformations;

(2) an ordering upon the variables consistent with the algebra (see (10)) and the $\star \circ \sigma$ automorphism;

we get the following q -mutation relations

$$\begin{aligned}\epsilon_L \epsilon_R &= q \epsilon_R \epsilon_L \\ \epsilon_L \theta_R &= q \theta_R \epsilon_L \\ \epsilon_L \theta_L &= q^2 \theta_L \epsilon_L \\ \epsilon_R \theta_L &= q^2 \theta_L \epsilon_R \\ \epsilon_R \theta_R &= q \theta_R \epsilon_R,\end{aligned}\tag{17}$$

and the FSUSY transformations of the fields Φ

$$\delta_\epsilon \Phi = \epsilon_L Q_L \Phi + \Phi Q_R \epsilon_R \quad (18)$$

or in components

$$\begin{aligned} \delta_\epsilon X &= q^2 \epsilon_L \psi_{20} + q \psi_{02} \epsilon_R \\ \delta_\epsilon \psi_{20} &= -q \epsilon_L \psi_{10} + q^2 \psi_{22} \epsilon_R \\ \delta_\epsilon \psi_{10} &= \epsilon_L \partial_z X + \psi_{12} \epsilon_R \\ \delta_\epsilon \psi_{02} &= q^2 \epsilon_L \psi_{22} - q^2 \psi_{01} \epsilon_R \\ \delta_\epsilon \psi_{01} &= q^2 \epsilon_L \psi_{21} + \partial_{\bar{z}} X \epsilon_R \\ \delta_\epsilon \psi_{22} &= -q \epsilon_L \psi_{12} - \psi_{21} \epsilon_R \\ \delta_\epsilon \psi_{11} &= \epsilon_L \partial_z \psi_{01} + q^2 \partial_{\bar{z}} \psi_{10} \epsilon_R \\ \delta_\epsilon \psi_{12} &= \epsilon_L \partial_z \psi_{02} - q \psi_{11} \epsilon_R \\ \delta_\epsilon \psi_{21} &= -q \epsilon_L \psi_{11} + q \partial_{\bar{z}} \psi_{20} \epsilon_R. \end{aligned} \quad (19)$$

The form of the action (16) is legitimated by the fact that the component $\theta_L^2 \theta_R^2$ transforms as a total derivative under FSUSY. Furthermore, the action is also invariant under conformal transformations.

IV. The Green Functions and the Wick contraction.

This section is devoted to the calculation of the Green functions associated to the action (16). Here, we will focus our attention on the holomorphic part of the action and we note X , ψ_1 , ψ_2 the basic fields. Two equivalent calculations will be proposed: the path integral approach and the mode expansion one. The latter will be useful for the normal ordering prescription and the operator product expansion (OPE) of the algebra.

IV.1 The Path Integral Approach.

We want to calculate the partition function Z

$$Z[0] = \int \mathcal{D}\psi_2 \mathcal{D}\psi_1 \exp \left[\int dz d\bar{z} \left(q \psi_2(z, \bar{z}) \partial_{\bar{z}} \psi_1(z, \bar{z}) - q^2 \psi_1(z, \bar{z}) \partial_{\bar{z}} \psi_2(z, \bar{z}) \right) \right] \quad (20)$$

Point out that the order of the path integration is opposite to the action one, in order to avoid the unwanted q -factor.

In (20), ψ_1 and ψ_2 are defined in the complex plane. In a discretization process, we just particularize the case where they are N -component vectors. On the same footing, the kinetic operator becomes a $N \times N$ matrix, noted A . So, we have to compute

$$Z[0] = \int (d\psi_2)^N (d\psi_1)^N \exp(\psi_1 A \psi_2) \quad (21)$$

It is known that any bilinear form can be diagonalized by two different transformations of determinant one, $\Delta = J A J'$. Using the property upon the integration on GGA

variables $\int (d\theta)^N = (\det J)^{-2} \int (d[J\theta])^N$ [15] (this can be seen directly from $\int d\theta = \frac{d^2}{d\theta^2}$, with an affine transformation) we get

$$Z[0] = \int (d\psi_2)^N (d\psi_1)^N \exp(\psi_1 A \psi_2) = \det(A)^2 \quad (22)$$

So, we obtain

$$\begin{aligned} Z[0] &= \int \mathcal{D}\psi_2 \mathcal{D}\psi_1 \exp \left(\int dz d\bar{z} \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} \begin{pmatrix} 0 & -q^2 \partial_{\bar{z}} \\ q \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} 0 & -q^2 \partial_{\bar{z}} \\ q \partial_{\bar{z}} & 0 \end{pmatrix}^2 \end{aligned} \quad (23)$$

Of course, the measure of integration has been defined in an appropriate way, such that the path integral (20) is just equal to $(\det A)^2$, in other words, each integral over ψ_1 and ψ_2 has been normalized by a $\frac{1}{\sqrt{2}}$ term and some phase factors. This result has been already obtained by Matheus-Valle *et al* in Ref. [13], and can be obviously extended for a GGA of any order ($\theta^n = 0$).

The two points Green function can be derived using the usual procedure, where two GGA sources are introduced (see for example Matheus-Valle *et al* in Ref.[13]). Here, we propose an alternative calculation with respect to the kinetic operator $A(z-w)$. The action can be rewritten in an equivalent way

$$S = \int d^2 z d^2 w \begin{pmatrix} \psi_1(z) & \psi_2(z) \end{pmatrix} A(z-w) \begin{pmatrix} \psi_1(w) \\ \psi_2(w) \end{pmatrix}$$

with,

$$A(z-w) = \begin{pmatrix} 0 & -q^2 \partial_{\bar{z}} \\ q \partial_{\bar{z}} & 0 \end{pmatrix} \times \delta(z-w) \delta(\bar{z}-\bar{w})$$

The propagator is then

$$\begin{aligned} \left\langle \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} \begin{pmatrix} \psi_1(w) & \psi_2(w) \end{pmatrix} \right\rangle &= \frac{\delta}{\delta A(z-w)} Z[0] \\ &= \begin{pmatrix} 0 & -q \\ q^2 & 0 \end{pmatrix} \frac{1}{z-w}. \end{aligned} \quad (24)$$

In this derivation, to avoid the unwanted 2 factor coming from the derivation of $(\det A)^2$, each fields is normalized with a $\frac{1}{\sqrt{2}}$ factor, as for the measure of integration. From (24) and from the well-known result on the propagator of scalar fields in $2D$, we can deduce the none-vanishing propagators

$$\begin{aligned} \langle X(z) X(w) \rangle &= (-\partial_z \partial_{\bar{z}})^{-1} = -\ln(z-w) \\ \langle \psi_1(z) \psi_2(w) \rangle &= \frac{q^2}{z-w} \\ \langle \psi_2(z) \psi_1(w) \rangle &= \frac{-q}{z-w}. \end{aligned} \quad (25)$$

From the propagator of ψ_1 and ψ_2 , it seems that they fulfill braiding properties, although they do not. This discrepancy will be explain further, in the next sub-section.

IV.2 The Modes Expansion

First of all, as we will justify in the next section, note that the fields ψ_1 and ψ_2 are respectively of conformal weight $2/3$ and $1/3$. Following the standard convention in Conformal Field Theory (CFT), their modes expansion can be expressed

$$\begin{aligned}\psi_1(z) &= \sum_{r_1 \in \mathbb{Z} + \frac{a}{3}} \psi_{1,r_1} z^{-r_1 - \frac{2}{3}} \\ \psi_2(z) &= \sum_{r_2 \in \mathbb{Z} + \frac{2a}{3}} \psi_{2,r_2} z^{-r_2 - \frac{1}{3}}\end{aligned}\tag{26}$$

By analogy with the string case, according to the value of a ($a = 0, 1, 2$), we will have different boundaries conditions ($z \rightarrow \exp(2i\pi)z$) for the ψ_i fields.

- For $a = 0$, ψ_1 picks up a q -phase factor and ψ_2 a q^2 one.
- For $a = 1$, ψ_1 and ψ_2 remain unaffected.
- For $a = 2$, ψ_1 picks up a q^2 -phase factor and ψ_2 a q one.

In these three situations, the Lagrangian remains, of course, unaffected. Point out that these sectors are adapted extension of the Ramond and Neveu-Schwarz [24] ones. Using the modes of the ψ fields, we can identify some of them with the θ_i or ∂_i . If one mode of ψ_1 is associated to θ , the corresponding mode of ψ_2 has to be associated to ∂ . Of course, it depends on the definition of the vacuum. Setting the following convention, associated to special choice of the vacuum, we obtain:

$$\begin{aligned}\psi_{1,r_1}|0\rangle &= 0, & r_1 > 0 \\ \psi_{2,r_2}|0\rangle &= 0, & r_2 > 0,\end{aligned}\tag{27}$$

and the q -mutation relations (which corresponds to the identification $\psi_{1,r} \equiv \theta_i$, $\psi_{2,r} \equiv \partial_{\theta_j}$ and the algebra (10-11)). Through this identification, we can in principle write all the q -mutation relations among the various modes. However, for our purpose we *only* need to know the q -mutations when the indices do not have same signs.

$$\begin{aligned}\psi_{2,r_2}\psi_{1,r_1} &= q^2\psi_{1,r_1}\psi_{2,r_2}, & r_2 < 0, & r_1 > 0, & r_2 \neq -r_1, \\ \psi_{2,r_2}\psi_{1,r_1} &= q\psi_{1,r_1}\psi_{2,r_2}, & r_2 > 0, & r_1 < 0, & r_2 \neq -r_1, \\ \psi_{2,r}\psi_{1,-r} - q\psi_{1,-r}\psi_{2,r} &= -q, & r > 0, \\ \psi_{2,-r}\psi_{1,r} - q^2\psi_{1,r}\psi_{2,-r} &= -q, & r > 0, \\ \psi_{2,r}\psi_{2,s} &= q\psi_{2,s}\psi_{2,r}, & r < 0, & s > 0, \\ \psi_{1,r}\psi_{1,s} &= q\psi_{1,s}\psi_{1,r}, & r < 0, & s > 0,\end{aligned}\tag{28}$$

those relations are very close to those obtained in one-dimension where quantization *à la Dirac* where used [15]. Notice that the derivative of θ is obtained using a change in the sign (see the third and fourth equations of (28)), in accordance with (26). From the choice of the vacuum (27) there is one and only one correspondence between the modes of ψ_1, ψ_2 and the generators of the algebra (10-11). It has to be emphasized, using (28), that nothing can be said on the q -mutation on the various fields but **ONLY** on their modes. In other words, nothing can be said on the symmetry of the wave function, but

only on the states in the Hilbert space. From the definition of the vacuum, it becomes now easy to derive the same propagator as before for the ψ 's fields. In the derivation of (24), using the modes expansion, one sees immediately the braiding property of the propagator because of the definition of the vacuum. This property is however lost in the general case: when we calculate $\langle \psi_1(z_1)\psi_2(z_2)\mathcal{O}(z_3) \rangle$ for an arbitrary operator \mathcal{O} .

To define the normal ordering, we proceed as usual putting to the right the creation operators and using explicitly the algebraic structure. From the identification

$$\begin{aligned}\psi_{1,r<0} &\sim a_{-r}^+ \\ \psi_{2,r>0} &\sim a_r \\ \psi_{2,r<0} &\sim b_{-r}^+ \\ \psi_{1,r>0} &\sim b_r\end{aligned}\tag{29}$$

the connection between the ψ variable and the q -oscillators can be found in Ref. [25]. It has to be stressed that such an identification enables an explicit construction of the modes of the fields (so the fields themselves) and their basic q -mutation relations, only from the coherence of the algebra (10-11).

Stress that, if nothing can be said on the q -mutation between the fields ψ_1 and ψ_2 , it is no more the case through a normal ordering. This is a consequence of the peculiar structure of the algebra (28). Indeed, we have

$$:\psi_1(z)\psi_2(z): = q^2 : \psi_2(z)\psi_1(z) : .\tag{30}$$

To get this equation we have used the definition of the vacuum, the q -mutation among the modes and a regularization of a term like $a_r^+ b_s^+$.

From this normal ordering prescription, if one wants to determine a 4-points Green function, using the Wick theorem, one is faced immediately with an ambiguity. In fact, if we calculate naively for instance $\langle : \psi_1(z)\psi_2(z) : : \psi_1(w)\psi_2(w) : \rangle$ we have two possibilities leading to different results :

- we can do first the contraction of $\psi_2(z)$ with $\psi_1(w)$, and then $\psi_1(z)$ with $\psi_2(w)$;
- we can q -mute the ψ 's inside the two normal ordering and then do the contraction of $\psi_1(z)$ with $\psi_2(w)$ and $\psi_2(z)$ with $\psi_1(w)$.

The result of those two different calculus will differ by a factor q ! This result never appears with fermions or bosons because the signs will compensate. This problem can be solved as follow:

Arguing that if $: AB : = q : BA :$, we do not have usually, after a transformation, $\delta(: AB :) = q\delta(: BA :)$. In order to be coherent, we have to impose for the variation of $: AB :$ the natural substitution $\frac{1}{2}(\delta(: AB :) + q\delta(: BA :))$. Paying attention that T and G are respectively the generators of the conformal and fractional supersymmetry transformations, we have to substitute in the two points Green functions involving T or G expressions analogous to

$$\frac{1}{2}(\langle : \psi_1(z)\psi_2(z) : : \psi_1(w)\psi_2(w) : \rangle + q \langle : \psi_2(z)\psi_1(z) : : \psi_2(w)\psi_1(w) : \rangle).$$

This procedure can be extended for N - points Green function using similar permutations.

Finally, before closing this section, we want to mention that, one can build propagator involving fractional power of $(z-w)$, using other algebraic structure than (10-11) or GGA. This is the essential difference of our result with respect to Saidi *et al.* This can be understood as follow[†]:

$$\frac{1}{(z-w)^\Delta} = \frac{1}{z^\Delta} \sum_{n \geq 0} a_n(\Delta) \left(\frac{w}{z}\right)^\Delta, \quad |z| > |w|.$$

If we change explicitly the q -mutations relations of $\psi_{1,r}$ with $\psi_{2,-r}$ by introducing on the RHS the appropriate numbers $a_n(\Delta)$ in Eq. (28), one can get

$$\langle \psi_1(z) \psi_2(w) \rangle \sim \frac{1}{(z-w)^\Delta}.$$

On the level of path integration, new rules have to be derived, substituting the result obtained in Eq. (22).

V. Current Algebra within Fractional Supersymmetry.

In this algebra, we have three different fields $X(z)$, $\psi_1(z)$ and $\psi_2(z)$ on which act two symmetries: the conformal and FSUSY transformations. The former will be generated by the stress momentum tensor $T(z)$ and the latter by the fractional supercurrent $G(z)$. In this section, as in the previous one, we consider only the holomorphic part.

V.1 The stress momentum tensor.

There are three ways to derive the stress momentum tensor $T(z)$.

1) First, coupling the different fields X, ψ_1, ψ_2 to the gravitational field, using the appropriate covariant derivative and invoking the standard general relativity results ([24], p.62).

2) The second, using Polyakov's results: coupling the fields in a non-conformally flat metric and performing an adapted transformation upon the variables ([26], p.236).

3) The third one is the most tractable for our purpose. From dimensional arguments, we deduce the conformal weights of the various fields. Using the definition of T as well as the transformation property of conformal field with conformal weight h , we get the expression of T . Let us detail this approach.

X is a conformal weight 0, so is Φ , the fractional superfield. $D_L^3 = -\partial_z$, so D_L, θ are of conformal weight $1/3$ and $-1/3$ respectively. It means that ψ_1 and ψ_2 are of conformal weight $2/3$ and $1/3$. So,

$$T(z) = -\frac{1}{2} : \partial_z X(z) \partial_z X(z) : + \frac{2}{3} q^2 : \psi_1(z) \partial_z \psi_2(z) : - \frac{1}{3} q^2 : \partial_z \psi_1(z) \psi_2(z) : \quad (31)$$

Note that T is of grade 0. Using the Wick theorem as well as the basic propagators, one can check

$$\begin{aligned} T(z)X(w) &= \frac{\partial_w X(w)}{(z-w)} + \dots, \quad |z| > |w| \\ T(z)\psi_1(w) &= \frac{\frac{2}{3}\psi_1(w)}{(z-w)^2} + \frac{\partial_w \psi_1(w)}{(z-w)} + \dots, \quad |z| > |w| \\ T(z)\psi_2(w) &= \frac{\frac{1}{3}\psi_2(w)}{(z-w)^2} + \frac{\partial_w \psi_2(w)}{(z-w)} + \dots, \quad |z| > |w| \end{aligned} \quad (32)$$

[†]This point has been mentioned to us by D. Bernard.

as it should be. The ... represents the regular part of the O. P. E.'s
 To prove the consistency of the algebra, the next point is to check the action of T on itself. After a little algebra, paying attention on the Wick contraction for 4-points Green function (see sect. IV.2), we have

$$T(z)T(w) = \frac{\frac{1}{2}(1 + \frac{2}{3})}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots, \quad |z| > |w| \quad (33)$$

In this O. P. E. the anomaly has two origins: one arising from the scalar field, as usual, and one from the ψ 's fields. We will come back to this point in the conclusion and outlooks. It has to be stressed that the action (16) leads naturally to a description of the fields with rational conformal weight (in this simplest case $1/3, 2/3$). So, the FSUSY transformation is a symmetry which connects states of spin $0, 1/3, 2/3$, generalizing, in that sense, the notion of supersymmetry. Before concluding this sub-section, we can say few words on the stress-momentum tensor. It is known that in $2D$, in addition to the invariance of the complex plane, one has the Weyl invariance acting on the metric: $g \rightarrow e^{\phi(z, \bar{z})}g$. With such a symmetry, plus the diffeomorphism $z \rightarrow f(z, \bar{z})$, $\bar{z} \rightarrow \bar{f}(z, \bar{z})$ we can globally eliminate the gravitation. In this special gauge, remains just the conformal symmetry which just transforms the metric up to a scale factor. So, the derivative has to be substituted by the appropriate covariant derivative (see *e.g.* [24], p.126). In this case we get

$$\begin{aligned} \nabla_z \psi_1(z) &= \partial_z \psi_1(z) - \frac{2}{3} \partial_z \phi(z), \\ \nabla_z \psi_2(z) &= \partial_z \psi_2(z) - \frac{1}{3} \partial_z \phi(z), \end{aligned}$$

with such a definition the stress-momentum tensor can be expressed with the normal or covariant derivatives because the Christoffel's symbols cancel.

V.2 The Fractional Supercurrent.

Using the results of Durand in Ref. [12], as well as the $2D$ -FSUSY transformations (19) we get

$$G(z) = -q^2 (: \partial_z X(z) \psi_2(z) : + \frac{1}{2} : \psi_1^2(z) :) \quad (34)$$

Along the same lines, as for the action of T on the fields, we can reproduce the FSUSY transformations on the fields

$$\begin{aligned} G(z)X(w) &= \frac{q^2 \psi_2(w)}{(z-w)} + \dots, \quad |z| > |w| \\ G(z)\psi_1(w) &= \frac{\partial_w X(w)}{(z-w)} + \dots, \quad |z| > |w| \\ G(z)\psi_2(w) &= \frac{-q \psi_1(w)}{(z-w)} + \dots, \quad |z| > |w| \end{aligned} \quad (35)$$

It has to be stressed that the action of the supercurrent on the fields gives the same transformation properties as in relations (19), as it should be. Now, it remains to check the closure of the algebra . We can calculate successively

$$\begin{aligned}
T(z)G(w) &= \frac{\frac{4}{3}G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{(z-w)} + \dots, \quad |z| > |w| \\
G(z)G(w) &= \frac{-q : \psi_2^2(w) :}{(z-w)^2} + \frac{\tilde{G}(w)}{(z-w)} + \dots, \quad |z| > |w|,
\end{aligned} \tag{36}$$

with $\tilde{G}(z) = (1 - q^2) : \partial_z X(z) \psi_1(z) : - q : \partial_z \psi_2(z) \psi_2(z) : .$ The first of these relation just tells us that G is a conformal field of conformal weight $4/3$. Now comes the question about $\tilde{G}(z) :$ is it a generator of a symmetry ? In other words can we find, in addition to the conformal and the FSUSY transformations, other symmetries of the complex plane.

Looking to the algebraic structure, one can see that those two symmetries are the only ones. In fact we cannot find symmetry with generators of grade 1 (the conformal/FSUSY transformations is generated by a grade 0/2 operator) *i.e* acting only on θ_L^2 and leaving θ_L unchanged. This result has also been proved on the level on the lagrangian by Matheus-Valle *et al* in [13]. Finally, using Durand's result derived in [12], *only* Q and Q^3 are generators of symmetry because they are the only ones that fulfill the Leibnitz rules. We will come back to this point in the next section. So, the new feature of these algebra, is that it will close under ternary and not bilinear identities. This is a reminiscence of the fact that the basic algebra (generated by Q_L and ∂_z , see sect. III) is not Lie or graded Lie algebra. Finally we can check the closure of the algebra.

$$\begin{aligned}
(G(z)\tilde{G}(w) + \tilde{G}(z)G(w)) &= \frac{(-2 + \frac{q}{2})}{(z-w)^3} - \frac{: \psi_2(w) \psi_1(w) :}{(z-w)^2} \\
&- \frac{6T(w) + : \psi_1(w) \partial_w \psi_2(w) : + : \psi_2(w) \partial_w \psi_1(w) :}{(z-w)}
\end{aligned} \tag{37}$$

In these relations, we have taken a symmetric product for G and \tilde{G} , in order to ensure the associativity of the algebra. To get this O.P.E. with have used $: \partial_z \psi_2 \psi_2 : = q^2 : \psi_2 \partial_z \psi_2 :$ arising from relation (28). The algebra has the special feature to close under ternary relations GGG , and with quadratic dependence on the fields. This algebra can be compared with the fractional superconformal algebra introduced in Ref. [6] which is also generated, in addition to the stress momentum tensor, by a current of conformal weight $4/3$. These two extensions of the Virasoro algebra are different. The fractional superconformal algebra closes with rational power of $z - w$, leading to non-local algebras because cuts are involved. The one we propose, closes only with *integer* power of $z - w$ but involves cubic relations instead of quadratic ones. In a similar way, we can mention that this feature is not specific to our model and already appears in the framework of the W_n algebra where polynomial dependence of the generators are involved to close the algebra [10]. However, invoking the remark done at the end of the previous section, by the appropriate substitution in Eq. (28), one should obtain the O.P.E. of the fractional superconformal algebra, *i.e.* with fractional powers of $(z - w)$. Of course, correlatively, the spin of the ψ_1 and ψ_2 fields change leading to different families of integrable models.

VI. Beyond 2d FSUSY.

In our paper, we have considered only two kinds of symmetries, one coming from the 2D conformal invariance and the second from the 2D FSUSY. We have argued that those symmetries were the only one to be considered in *our construction*. In this section, we are going to justify this point using algebraic arguments, and we will gauge the symmetries.

The fractional superfield Φ is defined in some appropriate representation of the fractional superspace $(z, \theta_L, \bar{z}, \theta_R)$. So, the symmetries of $(z, \theta_L, \bar{z}, \theta_R)$ acting on Φ are built up with the differential operators of the fractional superspace $(\partial_z, \partial_{\theta_L}, \delta_{\theta_L})$. For the sake of simplicity, we will consider in the following only the L -movers.

1) From the basic differential operators $\partial_z, \partial_{\theta_L}, \delta_{\theta_L}$, one can build *a priori* grade 0, 1, 2 operators. In this construction, because ∂_{θ_L} and δ_{θ_L} are of grade 2, only grade 2 and 0 operators can be built using first order differential operators (for example, Q and ∂_z , the generators of FSUSY and conformal transformations respectively belong to this category).

2) Arguing, that respectively Φ and $D\Phi$ have to transform in the same way, the covariant derivative has to commute with the generators of the symmetries. So, using the algebraic structure of GGA, the only allowed solutions are ∂_z, Q and Q^2 .

3) The third point, in this argumentation, implies that the product of two fractional superfields has to be also a fractional superfield. As Q^2 does not verify the Leibnitz rule, the once symmetries retained, in order to build invariant Lagrangian, are the FSUSY and the conformal transformations.

4) The remaining conserved Noether currents are T and G . In consequence, the would be conserved current associated to Q^2 , something like \tilde{G} defined in eq. (36), is not conserved and does not belong to the algebra. So, using the two generators T and G , we close the underlying basic symmetry using cubic relations.

Indeed, all these points can be extended for any F , F being the order of the FSUSY transformations.

Clearly, all these assertions are relevant within the framework of our kind of construction. Extension of the underlying symmetries in other approaches could in principle considered. This leaves the potentiality to add Q^2 as a generator of the associated symmetry. This peculiar property has been exploited in the second paper of [17] where a conserved spin 5/3 current were introduced in addition to a spin 4/3 current. Following [17] the algebra $\theta^3 = 0$ can be represented in a linear way introducing two Grassmann variables θ_1 and θ_2 satisfying

$$(\theta_1)^2 = \theta_2, \quad \theta_1\theta_2 + \theta_2\theta_1 = 0.$$

With this associated representation of the algebra, two generators (having a linear dependence in the previous variables and their derivatives) can be introduced. However, the possible representation consistent with the algebra (10-12) is not at first glance obvious and needs further investigations.

In addition to this discussion, we want discuss the basic points that lead, from the action defined in eq.(16) and invariant under global transformations, to an action invariant under Gauge symmetries. Of course, the full Lagrangian will not be exhibited, but only the relevant points dictated by the Noether procedure introducing Gauge fields that couple with their associated conserved current. The determination of the full invariant Lagrangian goes beyond the scope of this paper.

The $2D$ diffeomorphism (which contains the conformal transformations as a subgroup) are controlled by a metric or a “zweibein”. Similarly, the local FSUSY, *i.e.* the fractional supergravity (FSUGRA) can be controlled by a Gauge field analogous to the gravitino in supergravity, we have named fractino by analogy [15].

Due to the non-linearity of the algebra, the existence of one or two fractino(i) is still an open problem. In the first situation, we add to the Lagrangian, using Noether procedure, a term like

$$\chi_1 G + \dots, \quad (38)$$

where χ_1 defines a fractino and G the FSUSY current. Their spin are $4/3$, $-4/3$ respectively. In the second situation, the general Lagrangian has to be completed by an additive term including two fractini χ_1 and χ_2 :

$$\chi_1 G + \chi_2 \tilde{G} + \dots \quad (39)$$

Let us point out that \tilde{G} is the transformed field of G under FSUSY transformations: $\delta_\epsilon G = \epsilon \tilde{G}$. Arguing that ϵ is a spin $-1/3$ field, \tilde{G} and χ_2 are of spin $5/3$ and $-5/3$. The presence of \tilde{G} in (39) can be seen as a reminiscence of the peculiar structure of the algebra. Due to the fact that Q closes under a cubic power instead of a quadratic one, we might have two gauge fields instead of one as in the framework of Lie or super-Lie algebras. This second gauge field χ_2 is coupled to \tilde{G} , the would be conserved current of Q^2 (see point (4) of this section).

The various states of the spin for χ_1 , χ_2 are respectively $-4/3, -2/3, 2/3, 4/3$ and $-5/3, -1/3, 1/3, 5/3$ when the holomorphic and anti-holomorphic part of the action have been considered. This peculiar form of the projection can be explained as follow: χ_1 is a vector-spin $1/3$ field, $\chi_1 = \chi_{\pm 1, \pm \frac{1}{3}}$ and in an analogous manner $\chi_2 = \chi_{\pm 1, \pm \frac{2}{3}}$.

Using the Noether theorem, it is known that the fractino field χ_1 has to transform like:

$$\delta_\epsilon \chi_1 \sim \partial_x \epsilon.$$

where x stands respectively for z or \bar{z} according to the value of the spin of the fractino χ_1 . Those results goes along the same line as in superstring theory for the transformation law of the gravitino see ([24], p. 233). With the same arguments as those employed in $2D$ SUGRA and in connection with eq. (19), among the “zweibein”, χ_1 and χ_2 (which belong to the same fractional superfields), only the transformation of χ_1 involves derivatives.

In our construction, we can conclude that in addition to the conformal ghosts associated to the conformal transformations of the “zweibein”, only the FSUGRA ghosts associated to the χ_1 transformation has to be considered. This is because the *only* symmetries are generated by ∂_z and Q . We will come back to this point in the conclusion.

VII. Conclusion and outlooks.

We have obtained new structures extending the Virasoro algebra by considering a generalization of conformal symmetry. This symmetry, as we have seen, is a transformation which connects states of fractional spin. It has to be stressed that those algebras are not constructed from Lie or graded-Lie algebra; meaning that they do not close via quadratic relations. Consequently, we have obtained conformal field theory

which does not belong to the well-known model (fractional superconformal invariance) where fractional spin are involved [6].

The conformal dimension of our CFT is $5/3$. Taking into account this peculiar situation with $c = 5/3$, new $2D$ -integrable models could be described using $2D$ -fractional supersymmetry. Various extensions can be derived in this formalism. First of all, we can modify the q -mutation relations (28) leading to fractional two points functions and to conformal fields with other conformal weight. Clearly, their associated central charge will change. Secondly, instead of considering only a scalar superfield, we can introduce a fractional superfield of conformal weight h . Thirdly, it has been pointed out in [17] that representations of the FSUSY algebra can be obtained from the periodic representation of the quantum group $U_q(sl(2))$. Finally, one can introduce interactions, including a superpotential [N. Debergh in [12]], or a coupling between superfields of different conformal weight [Colatto *et al* in [12]].

In the context of string, if FSUSY is the symmetry of the worldsheet, this symmetry could be used in order to build solutions with relevant phenomenology (appropriate gauge group, three families of massless quarks and leptons, space-time supersymmetry etc.). It remains, of course, an open question. In that direction, we can easily calculate the critical dimension. The conformal anomaly has two origins: one coming from the space-time degrees of freedom ($c_{X,\psi_1,\psi_2} = 5/3D$, $5/3$ for each dimension), and the other coming from the ghost-part of the action. As in string theory, the conformal part is $c_{b-c} = -26$ [4, 24]. For the FSUSY part of the ghosts there is no need to know the specific part of the FSUSY-ghosts. From the conformal weight of ϵ_L (the parameter of the FSUSY transformations) and the transformations properties of GGA we get (with J , something like $\partial_{\bar{z}}$, the operator of the transformation on the gauge field which controls the local FSUSY, the fractino χ_1).

$$S_{FSUSY} = \det J^{-2} = \int \mathcal{D}\beta \mathcal{D}\gamma \mathcal{D}\beta' \mathcal{D}\gamma' \exp [S_{\beta,\gamma} + S_{\beta',\gamma'}],$$

where γ, γ' are two commuting ghosts of conformal weight $-1/3$ and β, β' two commuting ghosts of conformal weight $4/3$. Following the results obtained by Polyakov, the stress-momentum is known as well as the contribution to anomaly $c_{\beta,\gamma} = 2 \times 22/3$ ([26], p. 238). The critical dimension is then $D = \frac{34}{5}$.

If one builds a theory with Q^2 as a additional generator, one needs another pair of commuting ghosts of conformal weight $-2/3, 5/3$. Following the same procedure as before, one can show that their contribution to the anomaly is $2 \times \frac{23}{3}$, leading to a negative (!) critical dimension. So, in the context of string theory, the approach with *only* Q as a generator should be more appropriate. Of course, the critical dimension $D = \frac{34}{5}$ is meaningless, but for another F , appropriate integer dimension can eventually be reached.

Acknowledgments.

It is a pleasure to thank D. Bernard, N. Rivier, M. Rosso and J.B. Zuber for useful discussions and remarks.

Appendix

The purpose of this appendix is to construct explicitly the Klein transformation adapted to q -mutating numbers ($q = \exp(\frac{2i\pi}{3})$). Consider a set ξ_i, ∂_{ξ_i} , $i = 1 \dots p$ of commuting variables satisfying the following relations.

$$\partial_{\xi_i} \xi_i - q \xi_i \partial_{\xi_i} = 1 \quad (A.1).$$

For the sake of simplicity, we develop the method only for the derivative ∂ ; the case for the second derivative δ is totally similar. First, let us introduce a number operator N_i

$$N_i = \xi_i \partial_{\xi_i} + \frac{(1-q)^2}{(1-q^2)} (\xi_i)^2 (\partial_{\xi_i})^2 \quad (A.2),$$

fulfilling the commutation relations

$$[N_i, \xi_i] = \xi_i \quad (A.3)$$

$$[N_i, \partial_{\xi_i}] = -\partial_{\xi_i} \quad (A.4).$$

A direct calculation shows then that

$$\xi_i q^{N_i} = q^{N_i+1} \xi_i = q q^{N_i} \xi_i \quad (A.5)$$

$$\partial_{\xi_i} q^{N_i} = q^{N_i-1} \partial_{\xi_i} = q^{-1} q^{N_i} \partial_{\xi_i} \quad (A.6).$$

Introducing

$$\theta_i = \xi_i \prod_{j>i} q^{N_j} \quad (A.7)$$

$$\partial_i = \partial_{\xi_i} \prod_{j>i} q^{-N_j} \quad (A.8),$$

one can check that the θ, ∂ 's satisfy the correct algebra (10-11). So, we have built explicitly a cocycle, expressed in terms of the basic fields, and allowing a change in the statistics *i.e.* substituting commuting variables to q -muting ones. This is the principle of the Klein transformation. Obviously this can be extended in a similar way for any type of GGA ($\theta^n = 0$). Notice that a number operator has already been introduced by Durand in [12]. All these results can be easily found in the faithful matrix representation of Ref. [15].

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